Lesson 1
Introduction to Waves

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James Clerk Maxwell

Maxwell (Scottish) unified electricity and magnetism in 1864 with his now famous equations and showed that light is an electromagnetic wave

James Clerk Maxwell
(1831-1879)

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and c is the velocity of light in vacuum.

Outline

- What are waves?
  - 1-D wave
  - Phase, phase front, and phase velocity

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{E} - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

- Plane waves
- Why are EM waves transverse?
- Energy, Poynting vector, and intensity

- What are waves?
  - 1-D wave
  - Phase, phase front, and phase velocity

What are waves?

Anything that moves

Variable of both time and space

A way of energy delivery

What is a wave?

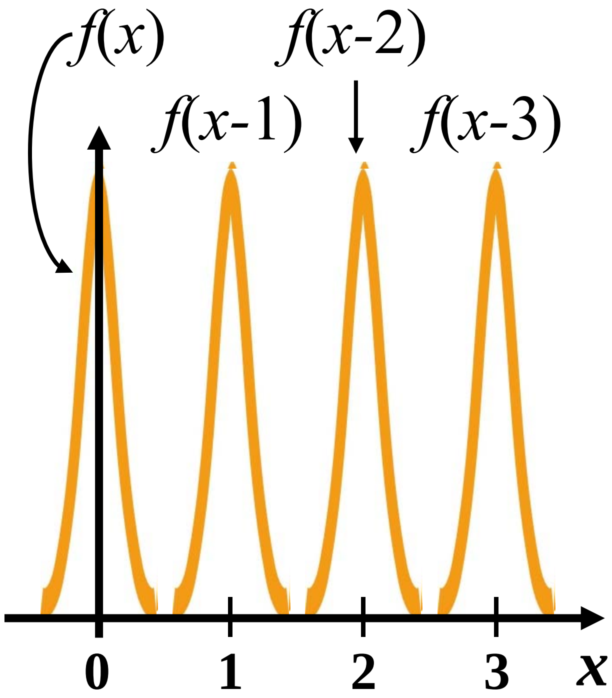
A wave is anything that moves.

Linear translation: $f(x)$ to $f(x\text{-}a)$.

If ***a*** **= v *t***, where **v** is positive and ***t*** is time, ➔time-dependent displacement.

So $f(x - \mathrm{v}\, t)$ represents a rightward, or forward, propagating wave.

Similarly, $f(x + \mathrm{v}\, t)$ represents a leftward, or backward, propagating wave.
v will be the velocity of the wave.



The 1-d wave equation and its solution

We already derived the wave equation.

Here is the one-dimensional form for scalar functions, f:

$$\boxed{\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} = 0}$$

The electric fields of light waves will be a solution to this equation. And v will be the velocity of light.

The wave equation has the simple solution:

$$\boxed{f(x,t) = f(x \pm \mathrm{v}t)}$$

where $f(u)$ can be any **twice-differentiable** function.

Proof that $f\,(x \pm \mathrm{v}t)$ solves the wave equation

Write $f(x \pm \mathrm{v}t)$ as $f(u)$, where $u = x \pm \mathrm{v}t$. So $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial t} = \pm \mathrm{v}$

Now, use the chain rule: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$ $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t}$

So $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}$ ➔ $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}$, and $\frac{\partial f}{\partial t} = \pm \mathrm{v}\frac{\partial f}{\partial u}$ ➔ $\frac{\partial^2 f}{\partial t^2} = \mathrm{v}^2\frac{\partial^2 f}{\partial u^2}$

Substituting into the wave equation:

$$\boxed{\frac{\partial^2 f}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} - \frac{1}{\mathrm{v}^2}\left\{\mathrm{v}^2\frac{\partial^2 f}{\partial u^2}\right\} = 0}$$

Simple solution to the wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Which has a simple sinusoidal solution:

$$\boxed{\vec{E}(\vec{r},t) = A\cos(\omega t \pm \vec{k}\cdot\vec{r} + \phi)\hat{e}}$$

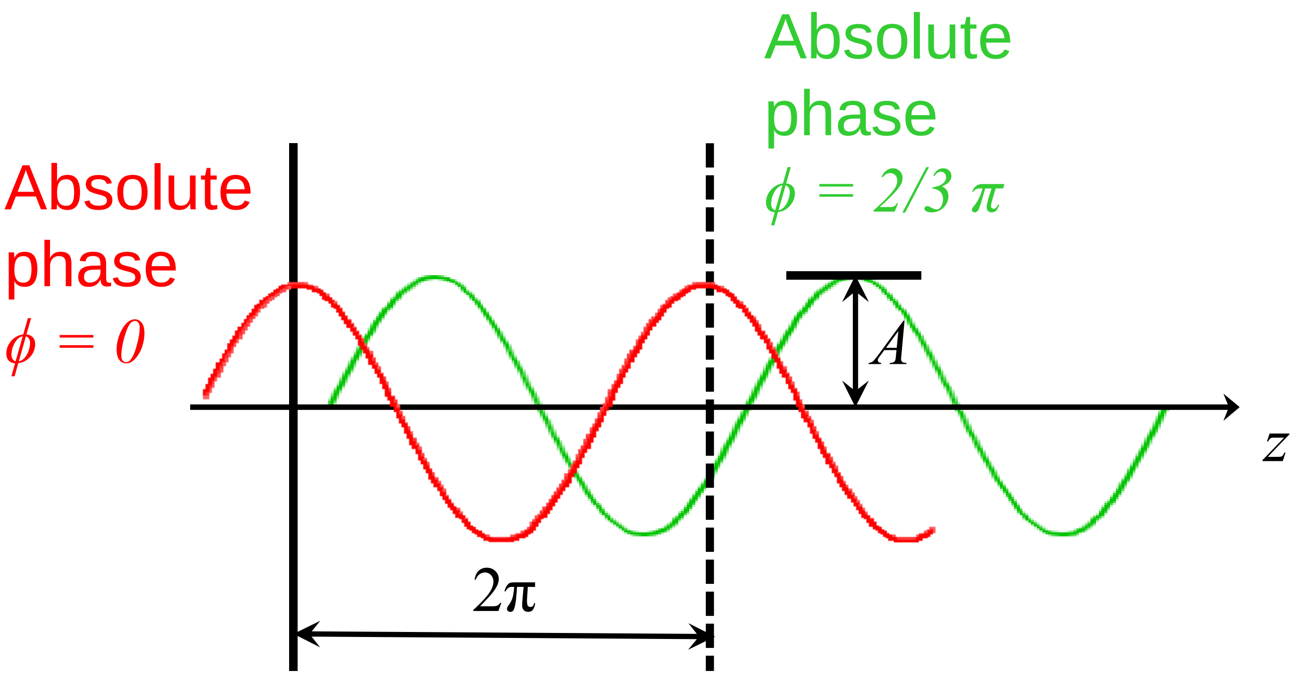
The same is true for the magnetic field.

Definitions: Amplitude and Absolute phase

$$E(z,t) = A\cos(\omega t - kz - \phi)$$

A = Amplitude

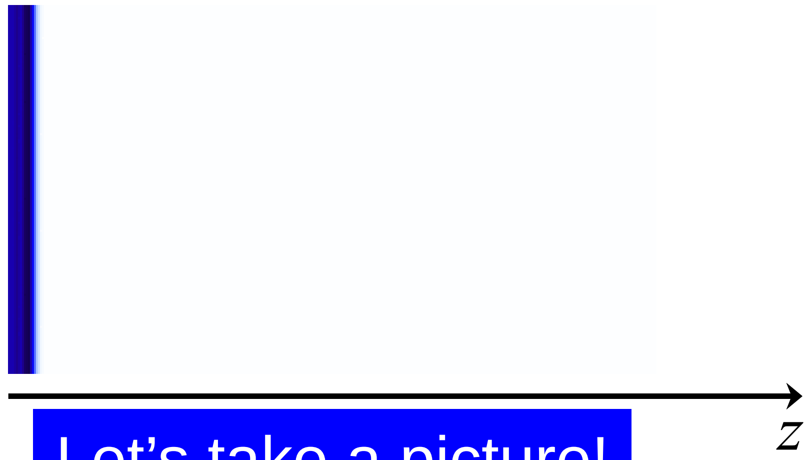
ϕ = Absolute phase (or initial phase)



Definitions

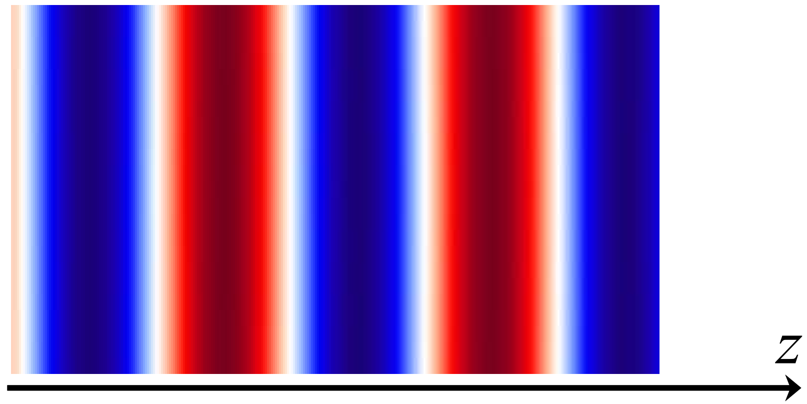
- Spatial quantities
  - **Wave/phase front**

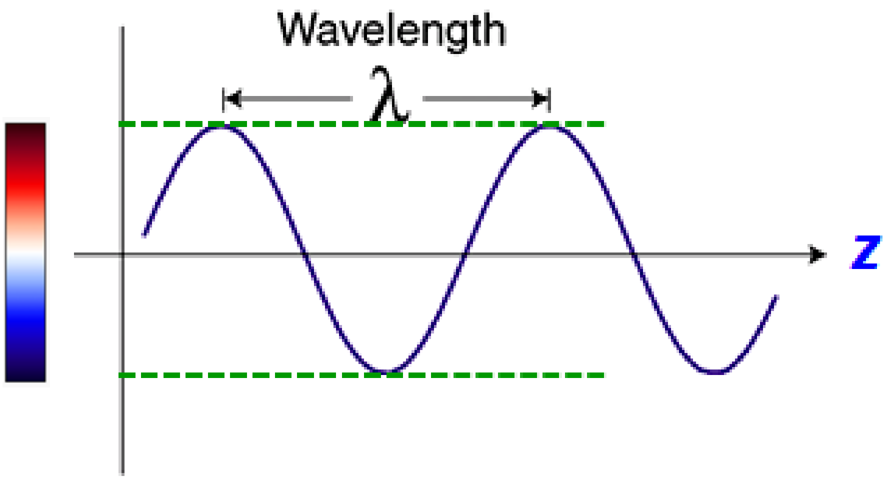
$$E(z,t) = A\cos(\omega t - kz)$$



Let's take a picture!
(Fixed t=0)

$$E(z,t) = A\cos(kz)$$

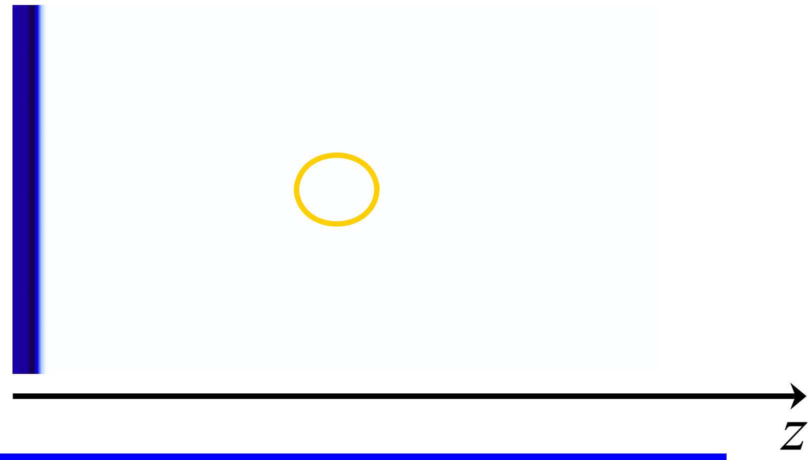




Definitions

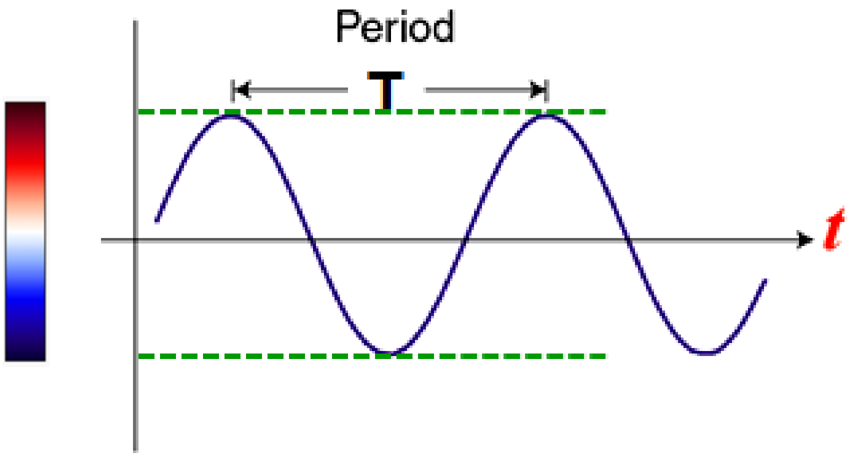
- Temporal quantities

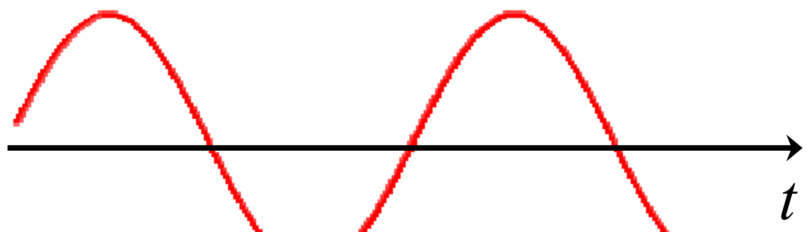
$$E(z,t) = A\cos(\omega t - kz)$$



Let's put a point detector!
(Fixed z=0)

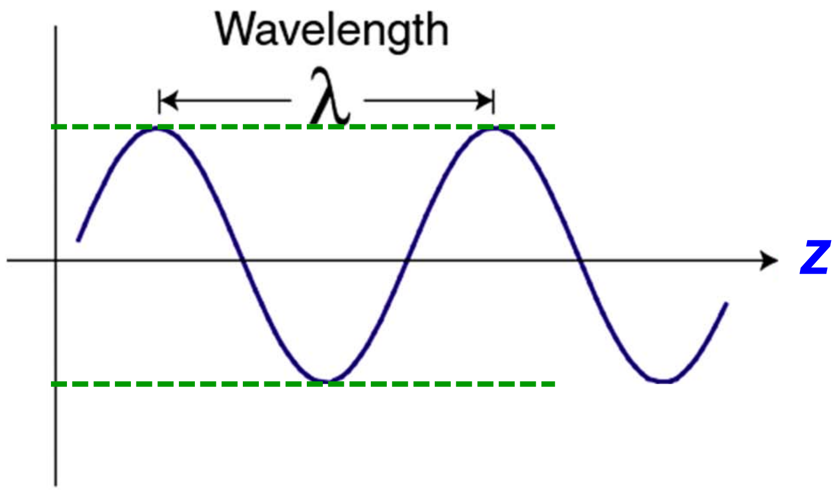
$$E(z,t) = A\cos(\omega t)$$



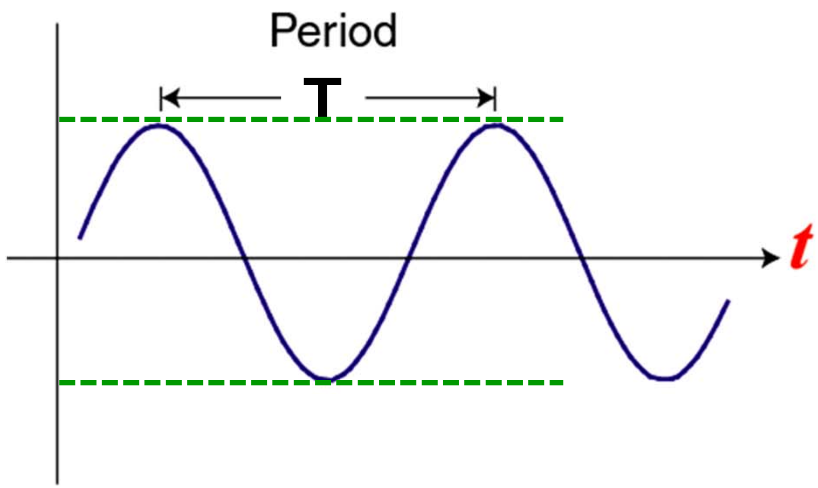


Definitions

- Spatial quantities
  - Wavelength λ
  - Wavenumber $\kappa=1/\lambda$
  - Propagation constant $k=2\pi/\lambda$



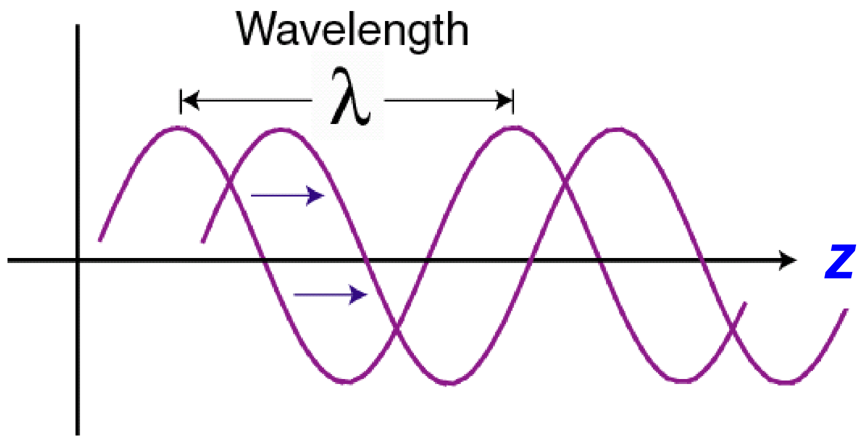
- Temporal quantities
  - Period T
  - Frequency f=1/T
  - Angular frequency $\omega=2\pi/T$



The phase velocity

How fast is the wave traveling?

Velocity is a reference distance divided by a reference time.



The phase velocity is the wavelength/period: $v = \lambda / T$

Since $f = 1/T$:

$$\boxed{v = \lambda f}$$

In terms of the k-vector, $k = 2\pi / \lambda$, and the angular frequency, $\omega = 2\pi / \tau$, this is:

$$\boxed{v = \omega / k}$$

- Plane waves
  - Definition
  - Field vs. propagation direction

The 3-D wave equation: vectorial!

A light wave can propagate in any direction in space. So we must allow the space derivative to be 3D:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

which has the solution: $\hat{E}(\vec{r},t) = \hat{E}_0 \exp^{j(\omega t - \vec{k}\cdot\vec{r} + \phi)}$

where

$$\vec{k} \equiv \left(k_x, k_y, k_z\right) \qquad \vec{r} \equiv \left(x, y, z\right)$$

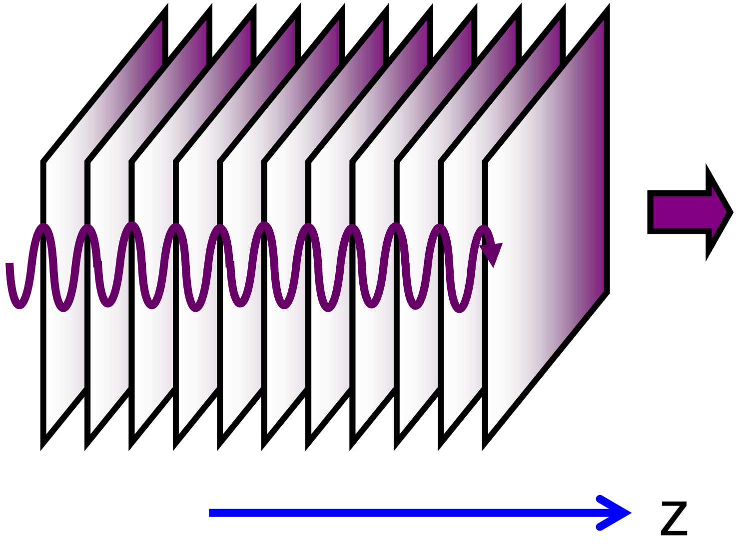
$$\vec{k}\cdot\vec{r} \equiv k_x\, x + k_y\, y + k_z\, z$$

and

$$k^2 \equiv k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu\varepsilon$$

Plane waves

- Uni-directional wave: assume propagating along $\hat{a}_z$
- Constant perpendicular phase front $E_x(z,t), E_y(z,t), E_z(z,t)$



- Vector Helmholtz's equation

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

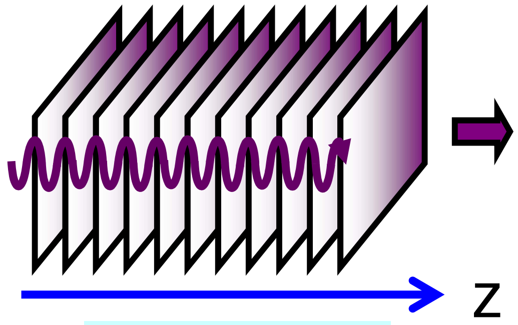
- Three ***scalar*** Helmholtz's equations

$$\begin{cases} \nabla^2 E_x + k^2 E_x = 0 \\ \nabla^2 E_y + k^2 E_y = 0 \\ \nabla^2 E_z + k^2 E_z = 0 \end{cases}$$

Solution to scalar Helmholtz's equation

- Only look at *scalar* Helmholtz's equation at E_x for example

$$\nabla^2 E_x + k^2 E_x = 0$$



$$E_x(z,t)$$

$$\left(\cancel{\frac{\partial^2}{\partial^2 x}} + \cancel{\frac{\partial^2}{\partial^2 y}} + \frac{\partial^2}{\partial^2 z} + k^2 \right) E_x = 0$$

$$\frac{\partial^2 E_x}{\partial^2 z} + k^2 E_x = 0$$

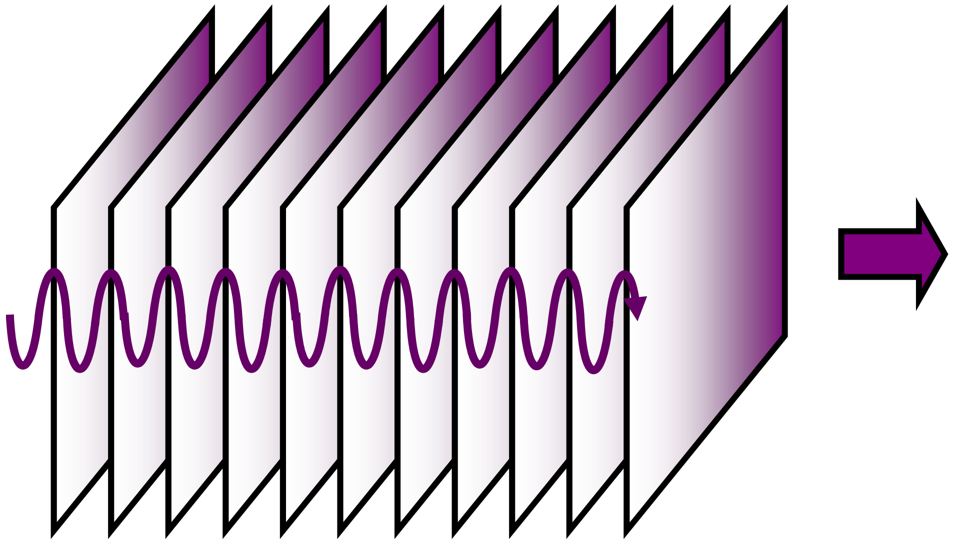
$$\Rightarrow E_x(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

Plane wave

$$\hat{E}(x,t) = \mathrm{Re}\{\hat{E}_0 \exp[j(\omega t \pm kz)]\}$$

A plane wave's contours of maximum field, called **wave-fronts** or **phase-fronts**, are planes. They extend over all space.

Wave-fronts are helpful for drawing pictures of interfering waves.



A wave's wave-fronts sweep along at the speed of light.

The wave-fronts are equally spaced, a wavelength apart.

They're **perpendicular** to the propagation direction.

Usually, we just draw lines; it's easier.

Wave that really moves!

- Plane wave
  - How to generate such movie?

Lot’s of knowledge behind these movies

- Why are EM waves transverse
  - E and B field orientation
  - Strength of magnetic field

Why are light waves transverse?

Suppose a wave propagates in the z-direction➔ $E_z(z,t)$

Then it's a function of z and t (and not x or y), so all x- and y-derivatives are zero:

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0$$

In vacuum: $\vec{\nabla} \cdot \vec{E} = 0$

$$\cancel{\frac{\partial E_x}{\partial x}} + \cancel{\frac{\partial E_y}{\partial y}} + \frac{\partial E_z}{\partial z} = 0$$

Substituting the zero values, we have: $\frac{\partial E_z}{\partial z} = 0$

So the longitudinal fields are at most **constant**, and not waves.

The magnetic field direction?

Suppose a wave propagates in the z-direction and has its electric field along the x-direction, so $E_y = E_z = 0$, and $E_x = E_x(z,t)$

What is the direction of the magnetic field?

Use: $$-\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} = \left(\cancel{\frac{\partial E_z}{\partial y}} - \cancel{\frac{\partial E_y}{\partial z}}, \frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}}, \cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}} \right)$$

So: $$-\frac{\partial \vec{B}}{\partial t} = \left(0, \frac{\partial E_x}{\partial z}, 0 \right)$$

In other words: $$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z}$$

the magnetic field is in the *y*-direction.

The strength of magnetic-field?

Start with: $-\dfrac{\partial B_y}{\partial t} = \dfrac{\partial E_x}{\partial z}$ and $E_x(z,t) = E_0 \exp\left[j\left(\omega t - kz\right)\right]$

We can integrate: $B_y(z,t) \;=\; B_y(z,0) \;-\; \displaystyle\int_0^t \frac{\partial E_x}{\partial z}\, dt$

Take $B_y(z,0) = 0$

So: $B_y(z,t) = \dfrac{jk}{j\omega} E_0 \exp\left[j(\omega t - kz)\right]$

But $\omega / k = v$: $\boxed{B_y(z,t) = \dfrac{1}{v} E_x(z,t)}$

The strength of magnetic-field?

- Use phasor-domain Maxwell's equations $\nabla \times \vec{E} = -j\omega\mu\vec{H}$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0\left(\cancel{\hat{a}_x H_x^+} + \hat{a}_y H_y^+ + \cancel{\hat{a}_z H_z^+}\right)$$

assume in air

$$H_y^+(z) = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z} = \frac{1}{-j\omega\mu_0}(-jk)E_x^+(z) = \frac{k}{\omega\mu_0}E_x^+(z)$$

Intrinsic impedance: $\eta_0 \equiv \dfrac{\omega\mu_0}{k_0} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377\ (\Omega)$ for free-space

$$\Rightarrow H_y^+(z) = \frac{1}{\eta}E_x^+(z)$$

Why we neglect the magnetic field

- The force on a charge, q, is: $\vec{F} = q\vec{E} + q\,\vec{v}_{particle} \times \vec{B}$

$$\vec{F}_{electrical} \qquad \vec{F}_{magnetic}$$

The ratio of the magnitudes of the two forces:

$$\frac{F_{magnetic}}{F_{electrical}} \leq \frac{q v_{particle} B}{qE}$$

$$\left|\vec{\mathrm{v}} \times \vec{B}\right| = \mathrm{v}B\sin\theta$$

$$\leq \mathrm{v}B$$

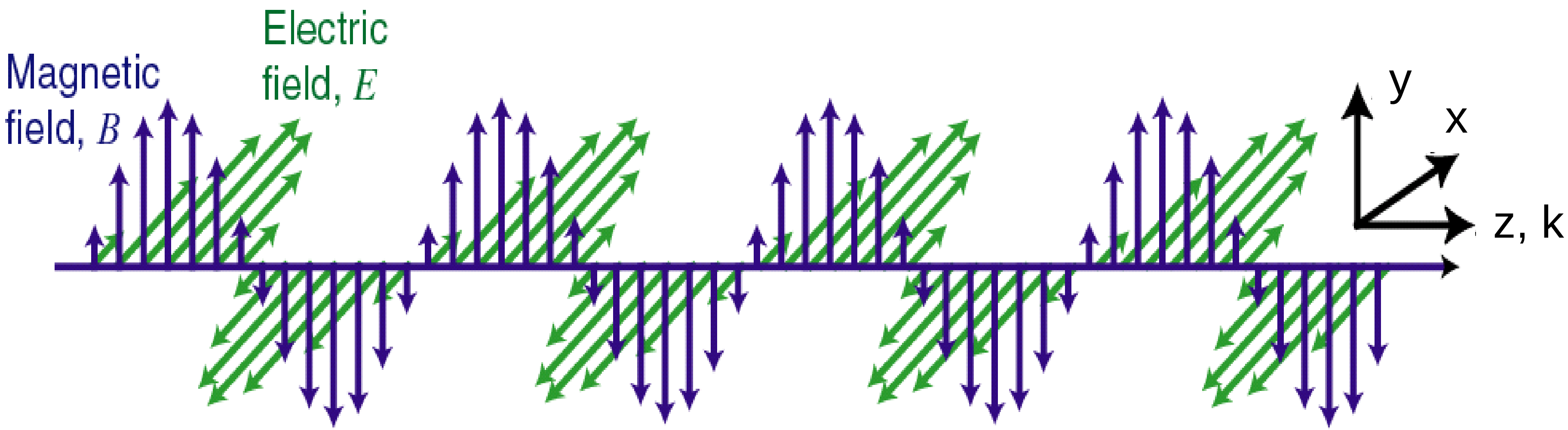
$$\Rightarrow \frac{F_{magnetic}}{F_{electrical}} \leq \frac{v_{particle}}{v_{light}}$$

So as long as a charge's velocity is much less than the speed of light, we can neglect the light's magnetic force compared to its electric force.

Light is transverse electromagnetic wave

The electric (E) and magnetic (B) fields are in phase.

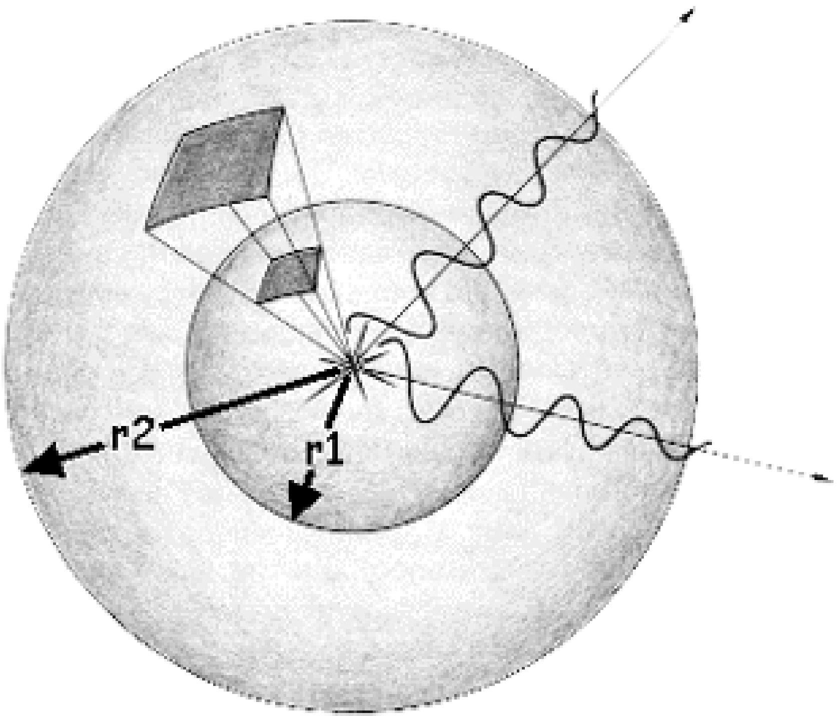
The electric field, the magnetic field, and the propagation direction are all perpendicular



Spherical wave

- Wave-front (phase front) that appear to be spherical

$$\hat{E}(x,t) = \frac{\hat{E}_0}{r}\exp[\,j(\omega t - \vec{k}\cdot\vec{r})]$$



- The energy of waves
  - Energy density
  - Intensity (irradiance)

The energy density of lightwave

The energy density of an electric field is: $U_E = \frac{1}{2}\varepsilon E^2$

The energy density of a magnetic field is: $U_B = \frac{1}{2}\frac{1}{\mu}B^2$

Using $B = E/v$, and $v = \frac{1}{\sqrt{\varepsilon\mu}}$, which together imply that $B = E\sqrt{\varepsilon\mu}$

Total energy density: $U_B = \frac{1}{2}\frac{1}{\mu}\left(E^2\varepsilon\mu\right) = \frac{1}{2}\varepsilon E^2 = U_E$

The electrical and magnetic energy densities in light are equal.

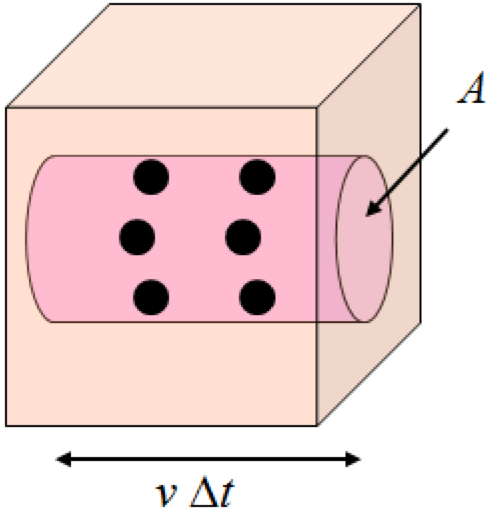
$$U = U_E + U_B = \varepsilon E^2$$

The Poynting vector

The power (instantaneous) per unit area in a beam.

Energy passing through area A in time Δt:

$$= U\ V = U\ A\ v\ \Delta t$$



So the energy per unit time per unit area:

$$= U\,V/(A\,\Delta t) = U\,A\,v\,\Delta t/(A\,\Delta t)$$
$$= U\ v = v\,\varepsilon\,E^2 = v^2\,\varepsilon\,E\,B$$

And the direction $\vec{E}\times\vec{B}\propto\vec{k}$ is reasonable.

$$\vec{E}=\hat{x}E_0\cos(\omega t-kz) \qquad \vec{B}=\hat{y}B_0\cos(\omega t-kz)$$

$$\Rightarrow \vec{S}=\hat{z}v^2\varepsilon E_0 B_0\cos^2(\omega t-kz)$$

$$\boxed{\begin{aligned}\vec{S}&=v^2\varepsilon\vec{E}\times\vec{B}\\ &=\vec{E}\times\vec{H}\end{aligned}}$$

The irradiance (often called the intensity)

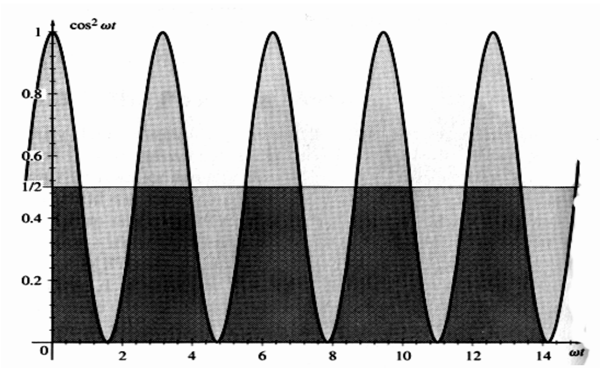
Intensity: A light wave's ***average*** power per unit area.

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{T} \int_{t-T/2}^{t+T/2} \vec{S}(\vec{r},t')\, dt'$$

Substituting a light wave for the Poynting vector, $\vec{S} = v^2 \varepsilon \, \vec{E} \times \vec{B}$

The average of $\cos^2$ is 1/2:

$$\vec{S}(\vec{r},t) = v^2 \; \varepsilon \, \vec{E}_0 \times \vec{B}_0 \cos^2(\omega t - \vec{k} \cdot \vec{r})$$



$$\Rightarrow \; I(\vec{r},t) \; = \left| \left\langle \vec{S}(\vec{r},t) \right\rangle \right| =$$

$$= \frac{1}{2} v^2 \; \varepsilon \left| \vec{E}_0 \times \vec{B}_0 \right|$$

The irradiance (continued)

Since the electric and magnetic fields are perpendicular and

$B_0 = E_0 / v, \quad I = \frac{1}{2} v^2 \, \varepsilon \left| \vec{E}_0 \times \vec{B}_0 \right|$ becomes:

$$\boxed{I = \frac{1}{2} nc \, \varepsilon_0 \left| \vec{E}_0 \right|^2}$$

where we used: $v = c / n$ and $\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0 n^2$

Remember: this formula only works when the wave is of the form:

$$\vec{E}(\vec{r}, t) = E_0 \exp\left[j\left(\omega t - \vec{k} \cdot \vec{r} \right) \right]$$

that is, when all the fields involved have the same $\omega t - \vec{k} \cdot \vec{r}$

Summary

- What are waves?
  - 1-D wave
  - Phase, phase front, and phase velocity

- Plane waves

- EM waves are transverse

- Energy, Poynting vector, and intensity

$$\boxed{I = \frac{1}{2} n c\, \varepsilon_0 \left| \vec{E}_0 \right|^2}$$